reasonably be expected, considering the *Radius* of the Instrument, and the Manner in which it was constructed.

THE PRINCIPLE OF LEAST ACTION.*

REMARKS ON SOME PASSAGES IN MACH'S MECHANICS.

Ernst Mach in his *Mechanics*[1] remarks,[2] with reference to the integral variational principles of Hamilton and of least action, that *other* such principles are possible, which idea has been suggestive to myself, and, as I have obtained some results which throw light on Mach's suggestions, I will try to describe the results here in not too technical language.[3]

I.

We must first of all notice a slight historical inexactitude in Mach's treatment of the principle of least action. "Maupertuis," we are told,[4] "enunciated, in 1747, a principle which he called '*le principe de la moindre quantité d'action.*'" Maupertuis[5] laid before the Paris Academy on April 15, 1744, a memoir in which he explained the reflection and refraction of light by a hypothesis substituted for Fermat's principle of least time.[6]

Maupertuis, like a good follower of Newton, accepted the emission hypothesis of light, and, according to P. Stäckel,[7] the contra-

* Philip E. B. Jourdain, an English scholar who has devoted his life to research in the line of modern logic, mathematics and pure mechanics, submits to us some remarks on Mach's *Science of Mechanics*. He is a devoted and zealous student of Mach's works and is as familiar with them as a theologian with his Bible. Being also well acquainted with the work of Georg Cantor, Peano and Bertrand Russell he is especially fitted to explain the theoretical aspect of pure mechanics. We are confident that his lucubrations serve a good purpose and therefore deem it wise to submit them to specialists by giving them space in our columns.

[1] *Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt*, 4th ed., Leipsic, 1901, pp. 395-413; Engl. transl. by T. J. McCormack under the title *The Science of Mechanics, a Critical and Historical Account of its Development*, 3d ed., Chicago, 1907, pp. 364-380. (This translation will be referred to as *Mechanics*, and the above German edition as *Mechanik*.)

[2] *Mechanik*, pp. 399, 402, 413; *Mechanics*, pp. 368-369, 371-372, 380.

[3] Cf. note on p. 78 of my paper "On the General Equations of Mechanics," *Quarterly Journal of Mathematics*, 1904, pp. 61-79.

[4] *Mechanik*, p. 395; *Mechanics*, p. 364.

[5] Cf. *Mechanik*, pp. 484-485; *Mechanics*, pp. 454-455.

[6] *Mechanik*, pp. 454-457; *Mechanics*, pp. 422-425.

[7] *Encykl. der math. Wiss.*, IV, I, (1908), p. 49, note 125. Stäckel wrongly refers to the *Berlin Mem.*, 1745, p. 276, for Maupertuis's application of the principle of least action to the motion of light.

diction that Mach found in Maupertuis's application of the principle of least action to the motion of light is due to Mach's mistaken supposition that Maupertuis worked on the basis of the undulatory theory.

On Fermat's principle of least time and Maupertuis's principle of least action, we will quote some passages from E. T. Whittaker's lately published book, *A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century.*[8]

"Descartes's theory of light rapidly displaced the conceptions which had held sway in the Middle Ages. The validity of his explanation of refraction was, however, called in question by his fellow-countryman Pierre de Fermat (b. 1601, d. 1665), and a controversy ensued which was kept up by the Cartesians long after the death of their master. Fermat[9] eventually introduced a new fundamental law, from which he proposed to deduce the paths of rays of light. This was the celebrated *Principle of Least Time,* enunciated[10] in the form, 'Nature always acts by the shortest course.' From it the law of reflection can readily be derived, since the path described by light between a point on the incident ray and a point on the reflected ray is the shortest possible consistent with the condition of meeting the reflecting surfaces.[11] In order to obtain the law of refraction, Fermat assumed that 'the resistance of the media is different,' and applied his 'method of maxima and minima' to find the paths which would be described in the least time from a point of one medium to a point of the other. In 1661 he arrived at the solution.[12] 'The result of my work,' he writes, 'has been the most extraordinary, the most unforeseen and the happiest, that ever was; for, after having performed all the equations, multiplications, antitheses and other operations of my method, and having finally finished the problem, I have found that my principle gives exactly and precisely the same proportion for the refractions which Monsieur

[8] London and Dublin, 1910, pp. 9-11, 102-103.

[9] *Renati Descartes Epistolae, Pars tertia;* Amsterdam, 1683. The Fermat correspondence is comprised in letters xxix to xlvi.

[10] *Epist.* xlii, written at Toulouse in August, 1657, to Monsieur de la Chambre; reprinted in *Œuvres de Fermat* (ed. 1891), Vol. II, p. 354.

[11] That reflected light follows the shortest path was no new result, for it had been affirmed (and attributed to Hero of Alexandria) in the *κεφάλαια τῶν ὀπτικῶν* of Heliodorus of Larissa, a work of which several editions were published in the seventeenth century.

[12] *Epist.* xliii, written at Toulouse on Jan. 1, 1662; reprinted in *Œuvres de Fermat,* Vol. II, p. 457; Vol. I, pp. 170, 173.

Descartes has established.' His surprise was all the greater, as he had supposed light to move more slowly in dense than in rare media, whereas Descartes had (as will be evident from the demonstration given above) been obliged to make the contrary supposition.

"Although Fermat's result was correct, and, indeed, of high permanent interest, the principles from which it was derived were metaphysical rather than physical in character, and consequently were of little use for the purpose of framing a mechanical explanation of light. Descartes's theory therefore held the field until the publication in 1667[13] of the *Micrographia* of Robert Hooke (b. 1635, d. 1703), one of the founders of the Royal Society, and at one time its Secretary."

Further on, we read (p. 102): ".... the echoes of the old controversy between Descartes and Fermat about the law of refraction were awakened[14] by Pierre Louis Moreau de Maupertuis (b. 1698, d. 1759).

"It will be remembered that according to Descartes the velocity of light is greatest in dense media, while according to Fermat the propagation is swiftest in free ether. The arguments of the corpuscular theory convinced Maupertuis that on this particular point Descartes was in the right; but nevertheless he wished to retain for science the beautiful method by which Fermat had derived his result. This he now proposed to do by modifying Fermat's principle so as to make it agree with the corpuscular theory; instead of assuming that light follows the *quickest* path, he supposed that 'the path described is that by which the quantity of action is the least'; and this *action* he defined to be proportional to the sum of the spaces described, each multiplied by the velocity with which it is traversed. Thus instead of Fermat's expression

$$\int dt \text{ or } \int \frac{ds}{v}$$

(where t denotes time, v velocity, and ds an element of the path) Maupertuis introduced

$$\int v \cdot ds$$

as the quantity which is to assume its minimum value when the path of integration is the actual path of light. Since Maupertuis's v, which denotes the velocity according to the corpuscular theory, is

[13] The *imprimatur* of Viscount Brouncker, P.R.S., is dated Nov. 23, 1664.

[14] *Mém. de l'Acad.*, 1744, pp. 417-426 [or *Œuvres de Mr. de Maupertuis*, Vol. IV, Lyons, 1756, pp. 3-18. To Maupertuis's work we will return on another occasion].

proportional to the reciprocal of Fermat's v, which denotes the velocity according to the wave-theory, the two expressions are really equivalent, and lead to the same law of refraction. Maupertuis's memoir is, however, of great interest from the point of view of dynamics; for his suggestion was subsequently developed by himself and by Euler and Lagrange into a general principle which covers the whole range of nature, so far as nature is a dynamical system."

* * *

In a memoir of 1746,[15] Maupertuis extended his hypothesis to all motions and called it the universal principle of rest and motion. By way of proving it, he derived the known laws of impact of inelastic and elastic bodies, and of the lever;[16] the motion of light having been dealt with in the memoir of 1744. It is most important to realize that, as A. Mayer[17] pointed out, Euler's discovery, made under the stimulus of the Bernoullis and published in the autumn of 1744 in an appendix to his *Methodus inveniendi,* was independent of Maupertuis, but that later on Euler's own tendency towards metaphysical speculation and the influence of Maupertuis combined to make Euler treat his principle in a less precise and more general way.

II.

Euler observed in 1744 that the differential equations of the motion of a particle are given by the simple requirement that the integral $\int v.ds$, where for the velocity v is substituted its value resulting from the principle of *vis viva,* and the integral is taken between two positions of the particle, should be a minimum. Euler

[15] "Les loix du mouvement et du repos déduites d'un principe métaphysique." *Mém. de l'Acad. de Berlin,* 1746, pp. 267-294. Voss (*Encykl. der math. Wiss.,* IV, 1, p. 95, note 256) has 1745 as the date of this memoir. This memoir was that analyzed by Mach (*Mechanik,* pp. 395-397; *Mechanics,* pp. 364-367). The analogies that exist between the motion of masses and the motion of light, which were noticed by Johann Bernoulli and by Möbius, were dealt with by Mach (*Mechanik,* pp. 402-408, 410-413, 457-459; *Mechanics,* pp. 372-380, 425-427). The principle of least action has been found very useful in optics, by Laplace, for example, in the treatment of astronomical refractions; and the mathematics of the theory of systems of rays built upon this one principle, which was the earliest work of William Rowan Hamilton, were later (in 1834 and 1835) transferred by Hamilton to the general problem of dynamics. Cf. P. Stäckel, *Encykl. der math. Wiss.,* IV, 1, 1908, pp. 489-493.

[16] In a memoir called "Loi du repos des corps" (*Mém. de l'Acad. de Paris,* 1740, pp. 170-176; *Œuvres,* Vol. IV, pp. 45-63) Maupertuis remarked that the work done when a final configuration of equilibrium is reached is generally either a maximum or a minimum (see Mach, *Mechanik,* pp. 69-75; *Mechanics,* pp. 68-73).

[17] *Geschichte des Prinzips der kleinsten Aktion,* Akademische Antrittsvorlesung, Leipsic, 1877; cf. my notes in *Ostwald's Klassiker,* No. 167, pp. 31-37.

expressly emphasized, first, that his theorem only holds if the principle of *vis viva* holds (and therefore cannot hold for motion in a resisting medium), and, secondly, that we must express v in terms of the attracting forces by quantities belonging to the orbit.[18]

Euler's work on this point was influenced adversely by his own tendency toward metaphysical speculation and Maupertuis's discovery—published some months before Euler's—of the obscure and almost theological universal "principle of the least quantity of action."[19]

III.

Lagrange[20] generalized Euler's theorem for the motion of any system of masses in the following way:

Let $m_1, m_2, m_3, \ldots$ be masses which act upon one another in any manner, and also, if we wish, move under the influence of any central forces which are proportional to any functions of the distances; let $s_1, s_2, s_3, \ldots$ be the spaces which are described by these masses in the time t, and let $v_1, v_2, v_3, \ldots$ be their velocities at the end of this time; then[21]

$$\Sigma m . \int v . ds$$

is a maximum or minimum, and thus, by the principles of the calculus of variations,

$$\Sigma m . \int (\delta v . ds + v . \delta ds) = 0. \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Lagrange eliminated the terms involving δv by making use of the equation

[18] Jacobi (see below), by direct generalization of Euler's theorem, reached his theorem.

[19] The early history of the principle of least action is very fully dealt with by me in my notes at the end of *Ostwald's Klassiker der exakten Wissenschaften*, No. 167.

[20] "Application de la méthode exposée dans le mémoire précédent à la solution de différents problèmes de dynamique," *Miscellanea Taurinensia* for 1760 and 1761, Vol. II, pp. 196-298; *Œuvres de Lagrange*, Vol. I, pp. 365-468. This memoir immediately followed Lagrange's first fundamental memoir on the calculus of variations: "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies," *Misc. Taur.*, 1760 and 1761 [published 1762], Vol. II, pp. 173-195; *Œuvres*, Vol. I, pp. 335-362; *Ostwald's Klassiker der exakten Wissenschaften*, No. 47, pp.3-30.
In Lagrange's first publication ("Recherches sur la méthode de maximis et minimis," *Misc. Taur.* for 1759, Vol. I; *Œuvres*, Vol. I, pp. 3-20), he announced (p. 15) his intention of deriving the whole of mechanics, by means of the principle of the least quantity of action, from a method he had of investigating the maxima and minima of indefinite integral formulae.

[21] For convenience of printing, the suffixes to the Σ, m, v, and s are here omitted. Instead of the now more usual Σ Lagrange (see below) used S.

$$\Sigma m.v.\delta v = \delta U \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

got by varying (differentiating with δ) the equation of *vis viva*.

Thus the equation (1), in conjunction with the condition (2), supposing that all the positions at the limits of the integral are given, so that there the variations of the coordinates are zero,[22] gives the fundamental equation[23]

$$\int S dm \{ (d\frac{dx}{dt} + \Pi dt)\delta x + \ldots \} = 0, \cdots\cdots\cdots\cdots (3)$$

where $$\Pi\delta x + \ldots = \delta U,$$

and S is a sign of a definite integral which refers to the masses of the system; so that, if there are a finite number of masses m_1, m_2, $m_3, \ldots$,

$$S dm = \Sigma m.$$

If there is an equation of condition $\phi = 0$ between the coordinates, the equation $\delta\phi = 0$ gives a relation between the δx's, δy's and δz's of (3); and then we can eliminate from (3) all of the variations except a certain number which is the degree of freedom of the system. If, then, we put the coefficient of every independent variation equal to zero, we obtain the necessary number of differential equations for the solution of the problem.

An important point is that, as Hölder[24] remarked, Lagrange[25] drew attention to the fact that, even when the expression for the element of work is not a complete differential, and consequently "δU" can only be regarded as an abbreviation, and not as a notation for the variation of a force-function, that the formula (2), or

$$\delta T = \delta U,$$

can be applied to get an extension of the principle of least action even to non-conservative forces. This wider form was not treated in Lagrange's later work in the *Mécanique analytique* on the principle of least action.

Thus Mach[26] is mistaken in stating that Lagrange "drew express attention to the fact that Euler's principle is applicable only in cases in which the principle of *vis viva* holds." Euler had already made this remark, and subsequently Jacobi strongly emphasized it; but Lagrange, correctly, as we now know, first drew attention to

[22] *Œuvres*, Vol. I, pp. 369-370.

[23] *Ibid.*, pp. 368, 406, 418, 435, 459.

[24] *Gött. Nachr.*, 1896, p. 136. In *Ostwald's Klassiker*, No. 167, last line on p. 39, for "Helmholtz" read "Hölder."

[25] *Œuvres*, Vol. I, pp. 384-385.

[26] *Mechank*, p. 401; *Mechanics*, p. 371.

the fact that the principle of least action, in the very general form which he gave it, does not depend for its validity on that principle of *vis viva,* which only follows from the general equations of mechanics under special conditions.

There was no mention of this extension in Lagrange's later works, and Hamilton, for example, only took from Lagrange the narrower form of the principle of least action which was given in the *Mécanique*.

* * *

Lagrange appears to have noticed that the integrand of (3), put equal to zero, is an expression of d'Alembert's principle; and, in that form, d'Alembert's principle is the fundamental formula of Lagrange's analytical mechanics,[27] and then the principle of least action became, for Lagrange, merely a result of the laws of mechanics, to be got by the integration of the simpler equation.

However, in the early memoir Lagrange had concluded from his generalized principle of least action nearly all the great results which later, in his *Mécanique,* he derived in another way; and so Jacobi[28] remarked that Lagrange's principle became the mother of our whole analytical mechanics.[29]

[27] D'Alembert's principle in combination with the principle of virtual displacements appeared in the above variational form for the first time in a prize essay of 1764 of Lagrange's on the libration of the moon (*Œuvres,* Vol. VI, pp. 5-61); and then, more fully, in a memoir of 1780 (*Œuvres,* Vol. V., pp. 5-122).

The various editions of Lagrange's *Mécanique* are: *Mécanique analitique,* Paris 1788, I vol.; second, greatly enlarged edition, *Mécanique analytique,* Paris, Vol. I, 1811, Vol. II (posthumous), 1815; third edition, with notes by J. Bertrand, 2 vols., Paris, 1853 and 1855; fourth edition, after the third, but with additional notes by G. Darboux, in *Œuvres de Lagrange,* Vols. XI, and XII, Paris, 1888 and 1889.

[28] See *Compt. Rend.,* Vol. V, 1837, pp. 61-67 (*Ges. Werke,* Vol. IV, pp. 129-136); *Vorlesungen über Dynamik, gehalten an der Universität zu Königsberg im Wintersemester* 1842-1843 *und nach einem von C. W. Borchardt ausgearbeiteten Hefte herausgegeben von A. Clebsch,* Berlin, 1866, p. 2 (2d ed., revised by E. Lottner, in Jacobi's *Ges. Werke, Supplementband*). Cf. A. Mayer, *Geschichte des Prinzips der kleinsten Action,* Leipsic, 1877, p. 26 (on Mayer's errors see my notes in *Ostwald's Klassiker,* No. 167).

In this early memoir the problems treated by Lagrange were: the motion of one body attracted by many fixed central forces; general problem of many attracting masses under any other forces; the finding of the orbits of two attracting bodies with respect to a third; a body in a plane under forces and drawing two other bodies by threads; a thread fixed at one end and charged with as many heavy bodies as wished; an inextensible thread, all the points being under any forces; the same problem with an extensible and elastic thread; motion of a body of any figure animated by any forces; laws of the motion of non-elastic and elastic fluids.

[29] However, Lagrange's method of multipliers (Mach, *Mechanik,* pp. 499-500; *Mechanics,* p. 471) appeared first in the *Méchanique analitique* of 1788.

After the publication of the *Mécanique,* the principle of least action fell into the background of interest until Hamilton, in 1834, showed that this principle had also a totally different title to our consideration. The only really important contribution to the exceedingly interesting questions that rise *à propos* of the principle of least action was an almost entirely neglected one made by Olinde Rodrigues in 1816.

IV.

In Lagrange's derivation, the variation of $v(=ds/dt)$ is not carried out, but the terms $m.v.\delta v$ are eliminated by the variational equation obtained from the principle of *vis viva.* Thus it is not necessary to decide whether t must be varied or not, whether we must put

$$\delta v=\frac{d\delta s}{dt}-\frac{ds}{dt}\frac{d\delta t}{dt} \text{ or } \delta v=\frac{d\delta s}{dt}.$$

It is almost beyond doubt that Lagrange would have maintained that the independent variable t was to be varied;[30] but Rodrigues was the first explicitly to say that, in this case, t must be varied.

Lagrange had worked with a *space* integral $\int\Sigma m.v.ds$, and had only remarked, in a short addition to the section on the principle of least action, made in the second edition of the *Mécanique,* that the above space integral transforms into the *time*-integral $\int 2\mathrm{T}.dt$, where 2T is the *vis viva* (or, as we now say, double the kinetic energy) of the system.[31] But Lagrange did not actually carry out the calculation of the variation of this time-integral; this was done by Rodrigues.[32] Rodrigues, as E. J. Routh[33] did later and apparently independently, to find the variation of $\int\mathrm{T}.dt$ under the condition $\mathrm{T}=\mathrm{U}+\text{const.}$ for the variation, so that $\delta\mathrm{T}-\delta\mathrm{U}=0$, multiplied the left-hand side of this last equation of condition by an undetermined factor, integrated it, added it to the variation of $\int\mathrm{T}.dt$, put all equal to zero, and then determined the factor.

[30] Cf. *Œuvres,* Vol. I, pp. 337, 345; and *Ostwald's Klassiker,* No. 167, p. 56.

[31] See *Ostwald's Klassiker,* No. 167, p. 11.

[32] *Correspondance sur l'Ecole polytech.,* Vol. III, 1816, pp. 159-162; German translation, with notes on some errors of Rodrigues, in *Ostwald's Klassiker,* No. 167, pp. 12-15, 41-42, 49-55.

[33] First in *An Elementary Treatise on the Dynamics of a System of Rigid Bodies,* 3d ed., London, 1877, pp. 305-312, 560-562. This passage coincides in essentials with *The Advanced Part* [Part II] *of a Treatise on the Dynamics of a System of Rigid Bodies,* 6th ed., London, 1905, pp. 301-309.

V.

The question as to whether the independent variable should be varied in the calculus of variations is of great importance to our conception of this calculus. According to Mach, [34] the first satisfactory explanation of the meaning of the process of variation used in this calculus was given by J. H. Jellett.[35] The value of the function $y = \phi(x)$ can vary by an (infinitesimal) increment dx of the independent variable, when we obtain the *differential*

$$dy = \phi(x + dx) - \phi(x),$$

or by the varying of the form ϕ of the function without x varying, so that $\phi(x)$ becomes

$$\phi_1(x) = \phi(x) + \epsilon\psi(x),$$

where ψ is an arbitrary function and ϵ is, for the definition of an *infinitesimal* variation, an infinitely small positive number. Then the *variation* of y is defined by

$$\delta y = \phi_1(x) - \phi(x).$$

Thus, if we keep, as is convenient, the term "variation" to denote alterations of value brought about by alteration of the form alone of the function, we seė that the independent variable is unaffected by our process of variation. On the other hand, Lagrange, as we have seen, held that the independent variable also was to be affected by the δ of the calculus of variations. Indeed, his claim that his method was more general than that of Euler rested partly on this ground. But other mathematicians appear mostly to have accepted that conception of a variation which Euler gave in a later memoir on Lagrange's method, that a "variation" of a function is brought about by a change in value of the constants occurring in that function. Thus, Jacobi, in his *Vorlesungen über Dynamik*,[36] stated that the variations δq of the generalized coordinates q contain merely the changes in value of the q's which arise from changes in value of the arbitrary constants occurring in the q's. Accordingly, he maintained[37] that the independent variable is not to be "varied," so that $\delta t = 0$.[38]

[34] *Mechanik*, pp. 468-474; *Mechanics*, pp. 437-443.

[35] *An Elementary Treatise on the Calculus of Variations*, Dublin, 1850, pp. 1, 5-6. Cf. A. Kneser, *Lehrbuch der Variationsrechnung*, Brunswick, 1900, pp. 1-2.

[36] *Werke, Supplementband*, p. 145.

[37] *Ibid.*, pp. 50, 59, 146, 149.

[38] Cf. similar views on the nature of a "variation" with Euler, Lagrange, Lacroix, G. W. Strauch, M. Ohm, Cauchy, and Stegmann in I. Todhunter's

So Jacobi, in his *Vorlesungen über Dynamik*,[39] stated that, in the action integral $\int \Sigma m . v . ds$, the time must be eliminated by the principle of *vis viva*, and all be reduced to space-elements. This, as Mayer remarked in his tract of 1877, was required by Euler in the case considered by him. Thus Jacobi's[40] formulation of the principle of least action was: If two positions of the system are given (that is to say, if we know the values which, for $x = a$ and $x = b$, the remaining $3n - 1$ coordinates receive), and we extend the integral

$$\int \sqrt{2(U+h)}\ \sqrt{\Sigma m\ .\ ds^2}$$

to the whole path of the system from the first position to the second, then its value is a minimum for the actual path as compared with all possible (consistent with the conditions, if there be any, of the system) paths.[41]

Mayer, in his tract of 1877,[42] accepted Jacobi's view that $\delta t = 0$ and consequently that, by means of the principle of *vis viva*, we must reduce all the quantities in the integrand to quantities which refer to the path of the system; and that the theorem of least action without this condition is quite meaningless. Since Lagrange did not eliminate the time, Mayer[43] concluded that Lagrange's theorem was meaningless, and what Lagrange really meant by his theorem was what is known as Hamilton's principle. This view had been previously maintained by M. Ostrogradski.[44]

But, in a memoir of 1886 on the general theorems of the calculus of variations which correspond to the two forms of the principle of least action in dynamics, Mayer[45] remarked, on the variation

work *A History of the Progress of the Calculus of Variations During the Nineteenth Century*, Cambridge and London, 1861, pp. 2, 8, 11, 13, 17-20, 31, 377, 378, 402, 413, 480-481.

[39] *Werke, Supplementband*, p. 44; *Ostwald's Klassiker*, No. 167, p. 17 (on pp. 16-26 is a reprint of Jacobi's sixth and part of his seventh lecture, which relate to the principle of least action).

[40] *Werke, Supplementband*, p. 45; *Ostwald's Klassiker*, No. 167, p. 18 (cf. the note on p. 55).

[41] On the limitations to the minimum-condition, which were pointed out by Jacobi (cf. Mach, *Mechanik*, p. 401; *Mechanics*, p. 371) see *Werke, Suppl.*, pp. 45-49; *Klassiker*, No. 167, pp. 18-22, 58.

[42] See p. 24, and *Klassiker*, No. 167, p. 57.

[43] *Op. cit.*, p. 27.

[44] *Klassiker*, No. 167, pp. 57-58.

[45] "Die beiden allgemeinen Sätze der Variationsrechnung, welche den beiden Formen des Prinzips der kleinsten Aktion in der Dynamik entsprechen," *Berichte der math.-phys. Classe der Kön. Sächs. Ges. der Wiss. zu Leipzig*, Sitzung am 14. November 1886, Vol. XXXVIII, pp. 343-355. The first person correctly to show the importance of Rodrigues's memoir was Th. Sloudsky

of t with Rodrigues: "Now, from the point of view of dynamics, in which we only permit variations from the instantaneous position of the system under consideration, that is so very unusual that I did not think at all of this possibility in my earlier work. But as soon as we neglect a purely dynamical signification (*Deutung*), and vary, not only the coordinates, but also the time, immediately that point which always caused the greatest doubts in Lagrange's derivation becomes clear. It is explained, namely, how the equation of *vis viva,* if it is prescribed as an equation of condition, can yet leave the variations of the coordinates quite unlimited,[46] and we see then that Jacobi's assertion that we must necessarily eliminate the time from the action-integral by means of the theorem of *vis viva* is not so; that, besides Jacobi's principle, there is a second, equally justified form of the principle of least action; and that it is this second form, and not Hamilton's principle inaccurately formulated, which Lagrange proved correctly, though certainly not with his usual clearness.

We may here remark that Routh,[47] from 1877 onwards and apparently independently of Rodrigues, also varied t, "by the fundamental theorem in the calculus of variations," and derived the principle of least action as Rodrigues did.

If t is to be varied, we must regard it, according to the conception of a "variation" derived from Jellett, as a function of another variable, θ, so that $\delta\theta = 0$ but δt is not zero in general. This was done explicitly by Helmholtz[48] in 1887.

Helmholtz also stated the view that Hamilton's principle is a form of Lagrange's principle. The grounds for this view are, as I showed in 1908,[49] more clearly evidenced in an identity established by Réthy under certain restrictions.

VI.

We have dealt with the question as to the relation of the principle of least action to Hamilton's principle, and we have seen how Lagrange, by working with a form which only contained the time through the velocities, and in which the variations of the velocities

(1866); Bertrand, in his notes on Lagrange's *Mécanique,* mentioned Rodrigues, but put $\delta(dq/dt) = d\delta q/dt$.

[46] Cf. *Klassiker,* No. 167, pp. 43-44.

[47] Cf. *ibid.,* pp. 50-51.

[48] "Zur Geschichte des Prinzips der kleinsten Aktion," *Sitzungsber. der Berliner Akad.,* Sitzung vom 10. März 1887, pp. 225-236; *Wiss. Abh.,* Vol. III, pp. 249-263.

[49] *Math. Ann.,* Vol. LXV, pp. 514-516.

could be at once eliminated by means of the varied equation of *vis viva,* allowed it to remain doubtful whether t was to be varied in the principle of least action, or not. We have seen how this question has given rise to discussions and misunderstandings which are connected with the principle of the calculus of variations, in the works of Rodrigues, Jacobi, Ostrogradski, Routh, Mayer, Sloudsky, Bertrand, Helmholtz, and Réthy. We have seen, finally, that Lagrange had attained to a very general formulation of the principle of least action, in which the equation of *vis viva* does not hold, a force-function does not exist, and the equations of condition may depend explicitly on the time. Thus Lagrange's principle is far more general than Jacobi's.

Of late years, the occurrence of differential and non-integrable equations among the equations of condition of a problem has assumed great importance. This happens in certain cases of rolling motion, and systems with such equations of condition were called by Hertz *non-holonomous.* The question arises as to whether the principle of least action and Hamilton's principle can be so formulated as to apply to non-holonomous systems. We shall see that Otto Hölder first succeeded in formulating extended forms of both principles which were completely equivalent to d'Alembert's principle. There were, of course, several points not dealt with by Hölder on which it was essential to be quite clear. Thus, the process of "variation" used by Hölder was not always the one to which we are accustomed in the calculus of variations, and the transformation of the principles from rectangular coordinates—which alone were used by Hölder—to more general coordinates gives rise to interesting questions. However, it seems to me that we have now reached a certain degree of finality in all these subjects, and we will now present the researches whose object was to extend the principles, in their proper order, and, where necessary, comment on them.

VII.

The question as to the extent of the variational principles begins with the publication, in 1894, of Heinrich Hertz's posthumous *Prinzipien der Mechanik.*[50] "The application of Hamilton's prin-

[50] *Gesammelte Werke von Heinrich Hertz,* Vol. III, *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt* (edited by Ph. Lenard, with a preface by H. von Helmholtz), Leipsic, 1894; English translation by D. E. Jones and J. T. Walley under the title *The Principles of Mechanics,* London, 1899.

ciple," said Hertz,[51] "to a material system does not exclude fixed connections between the coordinates chosen, but it requires that these connections can be exposed mathematically by means of finite equations between the coordinates; it does not permit of such connections as can be expressed only by differential equations. But nature itself appears not simply to exclude connections of the latter kind; for they occur if, for example, three-dimensional bodies roll upon one another without slipping."

Hertz[52] called a material system *holonomous* if between possible positions all thinkable continuous passages are also possible. The name was chosen to indicate that such a system is subject to integral (ὅλος) laws (νόμος), while material systems in general are subject only to differential laws. If the differential equations of condition of a material system can all be integrated, the coordinates of every possible position must satisfy the finite equations. The differences between the coordinates of two neighboring positions therefore satisfy an equal number of homogeneous linear differential equations, and, since these latter cannot contradict the given differential equations (in equal number) of the system, they satisfy the latter too. Thus the displacement between any two possible positions is a possible displacement, and thus the system is holonomous. Inversely, if the system is holonomous, its differential equations of condition allow an equal number of finite or integral equations between the coordinates themselves.

VIII.

Here we may digress to remark that the fact that cases of rolling motion give rise to equations of condition which are not integrable was observed by Routh, Ferrers (1873), and C. Neumann (1888).[53] The usual form of Lagrange's equations then fails. Of the extensions, what I have called, in the paper just quoted, "Routh's form" is the most important form for our present purposes. It involves Lagrange's multipliers, and is the only form of equation valid for non-holonomous systems which can be got directly by development of one of the integral variational principles. In deducing equations of motion from, say, Hamilton's principle,

[51] *Werke*, Vol. III, pp. 22-25; *Principles*, pp. 19-21.

[52] *Werke*, Vol. III, articles 123, 132, and 133 (pp. 91, 95, and 96); *Principles*, pp. 80, 84-85.

[53] Cf. the note on p. 63 of my paper "On the General Equations of Mechanics," *Quart. Journ. of Math.*, 1904, pp. 61-79. Cf. the bibliography in P. Appell's little book on *Les mouvements de roulement en dynamique*, Paris, 1899.

we so to speak divide the material system into a holonomous and a non-holonomous part. Suppose there are $3n$ rectangular co-ordinates of the system, k finite equations of condition between these coordinates and the time, and l non-integrable equations of condition. We form our integral for a system with $3n-k$ degrees of freedom and then eliminate the l superfluous coordinates by Lagrange's method.

IX.

An important paper on the differential equations of mechanics was written by A. Voss[54] in 1884 and published in 1885. In this paper, the equations of condition were used in their differential form, and were not assumed to be integrable, although the problems of rolling motion which caused such equations to be considered were not mentioned. The part which especially concerns us here is where Voss uses Hamilton's principle for the introduction of more general coordinates. He says[55] that, with non-integrable equations of condition, "the transformation can no longer be reduced to a problem of variations properly so called, but the property of the system of differential equations of condition of being a complete one forms the necessary and sufficient condition for this."

X.

Hertz decided that his own fundamental laws[56] holds both for holonomous and non-holonomous systems, and that from this law result the principle of least action[57] and Hamilton's principle[58] only under a limitation to holonomous systems. But this contradicts the general conviction[59] that Hamilton's principle is merely a transformation of d'Alembert's principle, and that the latter holds generally, and is equivalent to Hertz's law.[60] Thus arose the questions as to whether the usual derivation of Hamilton's principle from that of d'Alembert requires any limiting supposition. This question was the origin of the researches of Otho Hölder.[61] The very kernel of

[54] "Ueber die Differentialgleichungen der Mechanik," *Math. Ann.*, Vol. XXV, 1885, pp. 258-286.

[55] *Ibid.*, pp. 263-264.

[56] *Werke*, Vol. III, art. 309, p. 162; *Principles*, p. 144.

[57] *Werke*, Vol. III, arts. 347-356, pp. 174-176; *Principles*, pp. 155-157.

[58] *Werke*, Vol. III, arts. 358-362, p. 177; *Principles*, pp. 158-159.

[59] See, for example, Mach, *Mechanik*, pp. 413-414; *Mechanics*, p. 381.

[60] *Werke*, Vol. III, art. 394, p. 186; *Principles*, p. 166.

[61] "Ueber die Principien von Hamilton und Maupertuis," *Nachr. von der königl. Ges. der Wiss. zu Göttingen, Math. phys. Klasse*, 1896, pp. 122-157.

Hölder's work is his conception of the "variation of the motion of a system;" this it was which allowed him to give such a wide extension to the principles of least action and of Hamilton, so that the reply to the above question is: If d'Alembert's principle holds generally, so also must that of Hamilton, in its completest form; but if we choose Hertz's view that the varied path be a possible one, we get the limitation denoted by him. Hölder's conception of a varied motion is, then, paradoxical in so far that this "motion" need not be a possible one,—need not satisfy the equations of condition. It is, in Hölder's own words, only a mathematical auxiliary conception.

With Hertz, Hölder understood by "the position of a system" the totality of the positions of the material points of the system; the motion consists in a continuous sequence of positions of the system, which are passed through in a definite way with the time. To vary this original motion, we first give every system-position a small displacement, so that a new continuous sequence of positions arises. If the original sequence gives one position twice, we have two positions covering one another which can naturally be displaced in different manners. The starting position A and the final position B are to be fixed, and we refer each position on the varied path to one on the actual path. This correspondence is necessary in order that we may put the variation of an integral taken along the original path equal to the integral of the varied elements. We coordinate the identical initial positions to one another, and similarly with the two final positions.

If we imagine both the actual[62] and the fictitious motion to begin simultaneously at A, then the systems need not arrive at B simultaneously. In this case the corresponding positions on the two paths cannot all be passed simultaneously, and if the passage from an actual position to the corresponding position on the fictitious path be denoted by δ, so that, if the position P is actually reached at the time t, the corresponding position $P+\delta P$ is reached at the time $t+\delta t$ and $\delta(dt)=d(\delta t)$.

Now, in the most general manner of variation of the motion, we can still choose the velocity at each point of the varied path. This must be infinitely little different from the velocity at the corresponding position of the actual path, but is otherwise arbitrary.

Hölder then found the expression for δT in rectangular coordi-

[62] Thus it is assumed that the mechanical problem has one solution and one only.

nates, t and dt being affected by the δ-process, integrated the identity for δT from t_0 to t_1 (the times when the system, in the actual motion, is at A and B respectively), and integrated by parts. Thus, two parts are obtained: one integrated, which vanishes, since the variations of the coordinates at A and B vanish; and the other unintegrated, and we see by d'Alembert's principle, that the integrand of the last integral can be put equal to δU where, as before, "δU" only denotes the variation of a force function U in special cases—provided that the variations of the coordinates represent *virtual* displacements of the system.[63] Thus Hölder obtained the result that, where the δ-process is a process of giving every position P between A and B a virtual displacement to $P+\delta P$, and the aggregate of positions $P+\delta P$ is conceived as a fictitious path, then the equation

$$\int\{2T.d\delta t + (\delta T + \delta U)dt\} = 0, \quad \dots\dots\dots\dots (4)$$

where the integral is to be taken between the limits t_0 and t_1, is equivalent to d'Alembert's principle.

We cannot too strongly emphasize the nature of this varied path of the system. It is not necessarily a path that the system, however constrained, could take; that is to say, the connections of the system might have to be distorted from point to point. The displacement δP must be virtual at the instant t, but the position $P+\delta P$ is "reached" by the system, supposed to "move" on a fictitious path in a perhaps impossible way, at the, in general different, time $t+\delta t$. In fact, the fictitious path is only a possible one, of course under new constraints, if the equations of the condition are independent of the time, and the system is holonomous.

This fictitious motion is a useful conception because it enables us to see exactly why Hertz, for example, rather naturally limited the scope of the principle of least action and Hamilton's principle to holonomous systems; and also it allows us to formulate these principles in a perfectly general manner. That the conception of a "variation" is not that of the calculus of variations did not escape Hölder. "At the first glance," he wrote,[64] "the conception is perhaps peculiar, and it has been already said to me that I have no problem of variation properly so called. But that does not concern me. I am only concerned with giving a clear signification to the variations of the coordinates and the time which at the same time is such that

[63] That is to say, displacements consistent with the equations of condition and possible, at the instant considered. Cf., for example, Mach, *Mechanik, p.* 58; *Mechanics,* pp. 49, 56.

[64] In a letter to me of Jan. 15, 1904; cf. *Quart. Journ. of Math.,* 1904, p. 75, last note.

the principles hold as generally as is possible." In conformity with this, Hölder spoke of an "altered" (*abgeänderte*) instead of a "varied" motion.

In the above general principle, we can, without detracting from the equivalence to d'Alembert's principle, specialize the variations. Two ways at once suggest themselves:

(1) We may determine that corresponding positions are to be passed at the same instant, so that $\delta t = 0$, then (4) becomes a generalized Hamilton's principle;

(2) We may determine the velocity at each point of the varied path by fixing that $\delta T = \delta U$, the variation of the time being, of course, not zero; that is to say, using a more restricted phraseology for this wider case, the total energy is constant in a variation; then (4) gives the principle of least action in its most extended form.

XI.

There is one rather important point upon which Hölder only touched very briefly. I mean the introduction of other more general coordinates into the development of equations of motion from the above principles. Voss attempted to do this in 1900, but, as I have shown, [65] he used a method previously used by Routh and Réthy, which preserved the strictly variational character of the δ-process used even when the equations of condition depend explicitly on the time. Thus Voss unintentionally abandoned Hölder's δ-process. The application of Hölder's process to the formulation of the principles in general coordinates was first carried out by myself in the above cited paper of 1904, and more clearly in a paper of 1908.[66] Mathematically speaking, this formulation is not quite so simple as some might suppose; but here we are only concerned with the advantages of Hölder's δ-process over the strictly variational process in the formulation of the principle of least action and Hamilton's principle. The abandonment of the strict conception of a variation may appear to be a disadvantage. But surely this is compensated by greater simplicity; while, in my case, when we come to deal with non-holonomous systems we must abandon this strict conception, as was pointed out—we have seen above—by Voss in 1884 and by others later in somewhat different forms.[67] Further, unless the

[65] *Math. Ann.*, Vol. LXV, 1908, pp. 517-525.

[66] *Math. Ann.*, Vol. LXV, 1908, pp. 525-527.

[67] C. Neumann (1888), Hertz (1894), Hölder (1896), and Appell (1898); see also Boltzmann, *Vorlesungen über die Prinzipe der Mechanik*, Teil II, Leipsic, 1904, pp. 30-34.

equations of condition do not contain the time explicitly, the form of Réthy and Voss requires a condition holding for δt at the limits of integration, whereas in Hölder's generalized principle of least action no such condition is required.

XII.

As we have said at the beginning, Mach has stated, with reference to the principles of least action and Hamilton, that *other* such principles are possible. In this connection there are two investigations to which we must refer. The first was by Voss[68] in 1901, and was inspired by Hölder's work. Voss remarked that if not only the coordinates, but also the time is varied in the most general manner, δt can always be determined subsequently so that if we put the variation of the integral of *any* function of the coordinates and velocities equal to zero, we get the equation of motion. The second was an attempt by myself[69] to solve the problem suggested by Mach, by determining *all the possible integral variational principles*. For this purpose I inquired what was the most general form of the integrand in order that the principle obtained hence should be equivalent to Routh's extension of Lagrange's equations. The result was to find that Hölder's principle (4) was the most general of its kind, and, as Hölder had remarked, his principle may be specialized into Hamilton's principle or the principle of least action. These two principles are, in fact, two special cases out of the manifold of the principles equivalent to d'Alembert's principle and derivable from (4) by determining δt generally in all possible ways.

But there is another aspect of the matter. We have taken Lagrange's equations, or rather Routh's extension of them, as fundamental. But there are other forms of the equations of mechanics involving other quantities than Lagrange's T and U, and which sometimes present advantages over Lagrange's.[70] From these other equations we can derive[71] other variational principles not contained in Hölder's form (4), but since the functions in the integrand now involve differential coefficients with respect to t of the second

[68] "Bemerkungen über die Prinzipien der Mechanik," *Sitzber. der math.-phys. Klasse der k. Bayer. Akad. der Wiss. zu München*, Vol. XXXI, 1901, pp. 167-182, especially pp. 171-175; *Encykl. der math. Wiss.*, IV, 1, 1901, p. 94.

[69] *Quart. Journ. of Math.*, 1904, pp. 76-78.

[70] Cf. my paper on "Alternative Forms of the Equations of Mechanics," in the *Quart. Journ. of Math.*, 1905, pp. 284-296.

[71] Cf. *Ibid.*, pp. 290-295.

order, we must determine the varied path so that not only the variations δq but also the differentials $d\delta q$ of these variations vanish at the limits of integration. Analogous conditions as to the paths arise, if the integrand contains higher differential coefficients than the second.

XIII.

A curious result,[72] by the way, is that if we vary the integral of action $\int 2\mathrm{T}.dt$, so that δx means, as with Hölder, a virtual displacement of x, and vary t, we get exactly the same result as if we had not varied t either in T or in dt: the extra terms we get from varying t happen to cancel one another. Hence the faulty derivation, which we sometimes see, of Hamilton's principle from the principle of least action leads to correct results. This derivation is: Since $\delta\mathrm{T} = \delta\mathrm{U}$, we have

$$\delta\int 2\mathrm{T}.dt = \int(\delta\mathrm{T}+\delta\mathrm{T})dt = \int(\delta\mathrm{T}+\delta\mathrm{U})dt = \delta\int(\mathrm{T}+\mathrm{U})dt.$$

It should be noticed that the extra terms above referred to cancel even if the equations of condition contain the time explicitly. Further, we have seen that the identification maintained by Helmholtz and Réthy of Hamilton's principle with the principle of least action depended on the equations of condition not containing the time explicitly; and that the other identifications were based on misunderstandings. Finally, we have seen how in Hölder's other work, the true relation of the principles became clear, and how, at the same time, the principle became generalized.

XIV.

This sketch of the development and gradual generalization of a small part of the theory of mechanics gives us food for meditation. It seems to be necessary, in order that it may be possible to state the principles in question quite generally, to make use of a paradoxical conception—the conception of a generalized, fictitious "motion." It would be easy to say that the principles are, by the laws of logic, valid only under certain conditions; hence the paradox when we attempt to widen those conditions. But the paradox is not logical; it is merely verbal. We speak of a fictitious "path" and "motion" merely for the sake of picturesqueness: a mathematician no more means to imply the existence, in a mystical region of thought, of an impossible and fictitious path or motion, than he means to imply anything more than striking analogies of expression when he speaks,

[72] *Quart. Journ. of Math.*, 1904, pp. 78-79.

in analytical geometry, of "imaginary intersections" or "circular points at infinity." No philosopher wishes to confute a mathematician because, in his technical language, the mathematician may assert that some "real" numbers are not "rational."

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NOTES ON THE CONSTRUCTION OF MAGIC SQUARES OF ORDERS IN WHICH n IS OF THE GENERAL FORM $4p+2$.

It is well known that magic squares of the above orders, i. e., $6^2, 10^2, 14^2, 18^2$, etc., cannot be made perfectly pandiagonal and ornate with the natural series of numbers.

Dr. C. Planck has however pointed out that this disability is purely arithmetical, seeing that these magics can be readily constructed as perfect and ornate as any others with a properly selected series of numbers.

In all of these squares n is of the general form $4p+2$, but they can be divided into two classes:

Class I. Where n is of the form $8p-2$, as 6^2, 14^2, 22^2 etc.

Class II. Where n is of the form $8p+2$, as 10^2, 18^2, 26^2 etc.

The series for all magics of Class I may be derived by making a square of the natural series 1 to $(n+1)^2$ and discarding the numbers in the middle row and column.

Thus, for a 6^2 magic the series will be:

1	2	3	—	5	6	7
8	9	10	—	12	13	14
15	16	17	—	19	20	21
—	—	—	—	—	—	—
29	30	31	—	33	34	35
36	37	38	—	40	41	42
43	44	45	—	47	48	49

The series for all magics of Class II may be made by writing a square of the natural numbers 1 to $(n+3)^2$ and discarding the numbers in the *three* middle rows and columns. The series for a 10^2 magic, for example, will be: